

# The contact Eden bracket and the evolution of observables

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- To illustrate how it can be applied to **compute the evolution of an observable**.
- To apply it to a particular class of observables; those that satisfy the **mechanical condition**.

# Ingredients «*constrained contact Hamiltonian system*»

- *Configuration manifold:  $(T^*Q \times \mathbb{R}, \eta_Q = dz - \theta_Q)$ , where  $\theta_Q$  is the canonical Liouville form on  $T^*Q \times \mathbb{R}$ .*

# Ingredients «*constrained contact Hamiltonian system*»

- A *Riemannian metric*  $g$  on  $Q$  and, the *Lagrangian* given by

$$L(v_q, z) = T(v_q) - V(q, z), \quad \forall (v_q, z) \in T_q Q \times \mathbb{R}.$$

Here,  $T(v_q) = \frac{1}{2}g(v_q, v_q)$  represent the *kinetic energy*, and  $V$  is the *potential energy*.

# Ingredients «*constrained contact Hamiltonian system*»

- The *Legendre transformation*  $FL : TQ \times \mathbb{R} \rightarrow T^*Q \times \mathbb{R}$ .

# Ingredients «*constrained contact Hamiltonian system*»

- *Hamiltonian function:  $H = E_L \circ FL^{-1} : T^*Q \times \mathbb{R} \rightarrow \mathbb{R}$ .*

# Ingredients «*constrained contact Hamiltonian system*»

- *Linear kinematic constraints (on the velocities):* Regular distribution  $\mathcal{D}$  on  $Q$ .

$$\mathcal{D} = \{v \in TQ \mid \Phi^a(v) = 0\}. \quad (1)$$

# Ingredients «*constrained contact Hamiltonian system*»

- *Constrained submanifold:  $\mathcal{M} \times \mathbb{R} = FL(\mathcal{D} \times \mathbb{R})$ .*

# Ingredients «*constrained contact Hamiltonian system*»

- *Reaction forces*: Regular distribution  $\mathfrak{D}^l$  on  $P$ , s.t.,

$$\mathfrak{D}^{l^\circ} = (\pi_{Q,\mathbb{R}})^* \flat_g (\mathcal{M}^\circ),$$

where  $\flat_g$  is the *musical isomorphism* associated to  $g$  and  $\pi_{Q,\mathbb{R}} : T^*Q \times \mathbb{R} \rightarrow Q$  is the natural projection..

## Constrained Hamiltonian Herglotz equations

$$\left\{ \begin{array}{l} \frac{dq^i}{dt} \\ \frac{dp_i}{dt} \\ \frac{dz}{dt} \\ \Phi^a (b_g(\dot{\xi})) \end{array} \right. \begin{array}{l} = \frac{\partial H}{\partial p^i} \\ = -\frac{\partial H}{\partial q^i} - p_i \frac{\partial H}{\partial z} - \lambda_a \Phi_i^a \\ = p_i \frac{\partial H}{\partial p_i} - H \\ = 0, \forall a \end{array} \quad (2)$$

where  $\mathcal{D}^\circ$  is (locally) generated by the basis of 1-forms  $\Phi^a = \Phi_i^a dq^i$ .

## Theorem

Let be a Hamiltonian function  $H : T^*Q \times \mathbb{R} \rightarrow \mathbb{R}$ , and a constraint manifold  $\mathcal{M} \subseteq T^*Q$  as above. Let  $X$  be a vector field on  $T^*Q \times \mathbb{R}$  satisfying the equations

$$\begin{cases} b_Q(X) - dH + (H + \mathcal{R}_Q(H)) \eta_Q \in (\mathfrak{D}^l)^\circ \\ X|_{\mathcal{M} \times \mathbb{R}} \in \mathfrak{X}(\mathcal{M} \times \mathbb{R}). \end{cases} \quad (3)$$

Then, the integral curves  $\xi$  of  $X$  are solutions of the constrained Hamiltonian Herglotz equations.

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In this case, the solution of Eq. (3) is called *constrained Hamiltonian vector field*, and will be denoted by  $X_{H,\mathcal{M}}$ .

# *Nonholonomic Eden bracket*

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Let us consider the projection  $\gamma : T^*Q \times \mathbb{R} \rightarrow \mathcal{M} \times \mathbb{R}$ . Then, it turns natural to define the following bracket of functions,

$$\{f, g\}_E = \{f \circ \gamma, g \circ \gamma\}|_{\mathcal{M} \times \mathbb{R}} \quad (4)$$

for all  $f, g \in C^\infty(\mathcal{M} \times \mathbb{R})$ . This bracket will be called **contact Eden bracket** following the spirit of the articles of Eden.

Denote by  $X_H$  to the Hamiltonian vector field associated to the *unconstrained Hamiltonian system*  $(T^*Q \times \mathbb{R}, \eta_Q, H)$ .

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### Theorem

$$T\gamma \left( X_{H|_{\mathcal{M} \times \mathbb{R}}} \right) = X_{H, \mathcal{M}} = T\gamma \left( X_{H \circ \gamma|_{\mathcal{M} \times \mathbb{R}}} \right)$$

## Theorem

The contact Eden bracket  $\{\cdot, \cdot\}_E$  and the usual nonholonomic bracket  $\{\cdot, \cdot\}_{H, \mathcal{M}}$  coincide, namely

$$\{f, g\}_E = \{f \circ \gamma, g \circ \gamma\}|_{\mathcal{M} \times \mathbb{R}}$$

## Proposition

The contact Eden bracket  $\{\cdot, \cdot\}_E$  satisfies the following properties:

i) Let  $f$  be a function on  $\mathcal{M} \times \mathbb{R}$ . Then,

$$X_{H, \mathcal{M}}(f) = X_H(f \circ \gamma)$$

ii) For any function  $f$  on  $\mathcal{M} \times \mathbb{R}$ , we have that,

$$\{H, f\}_E = \{H, f \circ \gamma\}.$$

## Mechanical condition

A smooth function  $f$  on  $T^*Q \times \mathbb{R}$  satisfies the *mechanical condition*, if,

$$df|_{\mathcal{M} \times \mathbb{R}} \in {}^\perp \mathcal{D}^{I^\circ}.$$

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Thus, we denote the vector subspace of observables satisfying the mentioned mechanical condition by  $\mathcal{C}_{Mec}^\infty(T^*Q \times \mathbb{R})$

## Theorem

For any two functions  $f, g \in \mathcal{C}_{Mec}^\infty (T^*Q \times \mathbb{R})$ , we have that,

i)  $X_{H, \mathcal{M}}(f) = X_H(f)$

ii)  $\{f, g\}_E = \{f, g\}$

## Theorem

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- i)  $X_{H, \mathcal{M}}(f) = X_H(f)$
- ii)  $\{f, g\}_E = \{f, g\}$

In other words, **the dynamic and the evolution of the observables in  $\mathcal{C}_{Mec}^\infty (T^*Q \times \mathbb{R})$  are unconstrained.**

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- To construct some kind of **reduction** of the system by the observables whose dynamics are unconstrained.
- To investigate the “**differential space**” underlying the space of observables whose dynamics are unconstrained.
- To **extrapolate** the result to other kind of geometries: symplectic, multisymplectic, etc...

# THANKS!

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 M. de León, M. Epstein, and V. M. Jiménez.


*Contact Hamiltonian and Lagrangian systems with nonholonomic constraints.*

World Scientific, Singapore, 2021.

 M. de León, M. Lainz, A. López-Gordón, and J. C. Marrero

*Nonholonomic Brackets: Eden Revisited.*

Geometric Science of Information, 2023.

 F. Cantrijn and M. de León and D. Martín de Diego and J. C. Marrero

*Reduction of constrained systems with symmetries.*

Journal of Mathematical Physics, 1999.